THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Problem Set 2: The Real Numbers

1. If $a, b \in \mathbb{R}$, prove the following.
   (a) If $a + b = 0$, then $b = -a$,
   (b) $-(-a) = a$,
   (c) $(-1)a = -a$,
   (d) $(-1)(-1) = 1$,
   (e) $-(a + b) = (-a) + (-b)$,
   (f) $(-a) \cdot (-b) = a \cdot b$,
   (g) $1/(-a) = -(1/a)$,
   (h) $-(a/b) = (-a)/b$ if $b \neq 0$.
2. If $a \in \mathbb{R}$ satisfies $a \cdot a = a$, prove that either $a = 0$ or $a = 1$.
3. If $a \neq 0$ and $b \neq 0$, show that $1/(ab) = (1/a)(1/b)$.
4. If a, b and c are real numbers, prove that
   (a) If $a < b$ and $c \leq d$, prove that $a + c < b + d$.
   (b) If $0 < a < b$ and $0 \leq c \leq d$, prove that $0 \leq ac \leq bd$.
5. (a) Show that if $a > 0$, then $1/a > 0$ and $1/(1/a) = a$.
   (b) Show that if $a < b$, then $a < \dfrac{1}{2}(a + b) < b$.
6. (a) Prove there is no $n \in \mathbb{N}$ such that $0 < n < 1$. (Use the Well-Ordering Property of $\mathbb{N}$.)
   (b) Prove that no natural number can be both even and odd.
7. Let $S_1 := \{x \in \mathbb{R} : x \geq 0\}$. Show in detail that the set S_1 has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.
8. Let $S_2 := \{x \in \mathbb{R} : x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.
9. Let $S_3 := \{1/n : n \in \mathbb{N}\}$. Show that $\sup S_3 = 1$ and $\inf S_3 = 0$.
10. Let $S_4 := \{1 - (-1)^n : n \in \mathbb{N}\}$. Find $\sup S_4$ and $\inf S_4$.
11. Let $S_5 := \{r \in \mathbb{Q} : r > 0\}$. Find $\inf S_5$
12. Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$.
13. Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that $\sup(A \sup B) = \sup\{\sup A, \sup B\}$. (Remark: Can it be generalized to a finite / an infinite collection of bounded subsets of $\mathbb{R}$?)

14. Let $S \subseteq \mathbb{R}$ and suppose that $s^* := \sup S$ belongs to S. If $u \notin S$, show that $\sup(S \sup\{u\}) = \sup\{s^*, u\}$. (Remark: Special case of the previous question with suitable modification.)

15. Show that a nonempty finite set $\sup S \subseteq \mathbb{R}$ contains its supremum. (Hint: Use Mathematical Induction and the preceding exercise.)

16. If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.

17. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup A + B = \sup A + \sup B$ and $\inf A + B = \inf A + \inf B$.

18. If $y > 0$, show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$.

19. If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (Hence the set $\{ry : r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.)

20. Suppose a and b are positive real numbers. Show that there exists $n \in \mathbb{N}$ such that $a/n < b$.

21. Let $I_n := [0, 1/n]$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

22. Let $J_n := (0, 1/n)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} J_n = \phi$.

23. Let $K_n := (n, \infty)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} K_n = \{0\}$.

24. Suppose I_n, $n \in \mathbb{N}$, is a nested sequence of closed bounded intervals. Prove that I_{2n}, $n \in \mathbb{N}$, is also a nested sequence of closed bounded intervals. Furthermore, if $\xi \in \mathbb{R}$ such that $\xi \in I_{2n}$ for any $n \in \mathbb{N}$, show that $\xi \in I_n$ for any $n \in \mathbb{N}$.